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Existence results of Fractional Hilfer Boundary Value Problem Involving Nonlocal Integral Boundary Conditions

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Abstract

The existence and uniqueness (EU) results for Boundary value problem (BVP) with Hilfer fractional derivative (HFD) comprising integral nonlocal boundary condition(BC) are the focus of this study. To arrive at our desired findings, we have engaged the LeraySchauder nonlinear alternative and the fixed-point theorem by Boyd and Wong on nonlinear contractions. Additionally, examples have been provided to highlight our findings.

Keywords: Nonlocal Integral BC; HFD; BVP; Fixed Point Theorems; EU, 2020 MSC Classification: 26A33; 34B15.

1. Motivation of the study

In literature,the differential equations of fractional order have attracted more attention of the reader because of its numerous applications rather than integer order differentials. They have been proved very helpful in modeling aspect in several fields of science and technology. They have many numerous applications in control theory, memory and hereditary processes, viscoelasticity etc.(see[3,13,14,15,20,26]). Various fractional derivatives like RLFD, CFD, HFD, Hadamard, Hilfer-Hadamard, Katugampola, Miller-Ross etc. have been considered by several researchers (see[4,6,16,17,18,19,22,23]). HFD was introduced by R.Hilfer(see[10,11,12]). A lot of work on IVPs involving HFD have been done by many authors (see[7,8,25]). But there is less work on BVP. Inspired by the work done in literature on nonlocal BVP involving HFD (see[1,5,24]), in this paper we will establish existence and uniqueness results for solution of following BVP involving HFD with nonlocal integral BC as

$$\begin{cases} {}^{H}D^{\zeta,\alpha}\chi(\mathfrak{z}) + \wedge(\mathfrak{z},\chi) = 0, \mathfrak{z} \in [\mathfrak{p},\mathfrak{q}], \mathfrak{p} \geq 0, \\ \chi(\mathfrak{p}) = \chi'(\mathfrak{p}) = 0, \chi''(\mathfrak{p}) = 0, \chi(\mathfrak{q}) = k\int_{\mathfrak{p}}^{\tau} \chi(s)ds, \end{cases} \quad (1)$$

where $\chi \in \mathcal{C}^4([\mathfrak{p},\mathfrak{q}],\mathbb{R})$, $3 < \zeta \leq 4$, $0 \leq \alpha \leq 1$, $\tau \in (\mathfrak{p},\mathfrak{q})$, $k \in \mathbb{R}$, $\wedge: [\mathfrak{p},\mathfrak{q}] \times \mathbb{R} \to \mathbb{R}$ is a function such that $\wedge(\mathfrak{z},0) \neq 0$, ${}^{H}D^{\zeta,\alpha}$ represents HFD of order ζ and parameter α.
We will use the Leray Schauder nonlinear alternative and the Boyd and Wong fixed point theory on nonlinear contractions to demonstrate our findings.

Remaining sections of the article are organised as follows: In section 2, we review fundamental principles of fractional calculus that were utilised in this work. The qualitative results concerning existence and uniqueness results are the primary focus of section 3. Section 4 comprises some examples authenticating the established findings followed by the conclusion of this study.

2. Preliminaries

With the aim of the advancement of analysis this section outlines certain fundamental concepts in fractional calculus and its related properties with appropriate explanations.

Definition 1.[13] "RLFI of order $\zeta > 0$ for a continuous function $\chi: (\mathfrak{p}, \infty) \to \mathbb{R}$ is defined as $I^{\zeta}\chi(\mathfrak{z}) = \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z} - s)^{\zeta-1}\chi(s)ds,$ (2)

provided the integral converges at the right sides over $(\mathfrak{p}, \infty), \mathfrak{p} \geq 0$. "

Definition 2.[13] "RLFD of order $\zeta > 0$, for a function $\chi \in \mathcal{C}^{n}((\mathfrak{p}, \infty), \mathbb{R})$, $\mathfrak{p} \geq 0$ is defined as ${}^{RL}D^{\zeta}\chi(\mathfrak{z}) = \frac{1}{\Gamma(n-\zeta)}\frac{d^n}{d\mathfrak{z}^n}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z} - s)^{n-\zeta-1}\chi(s)ds, n-1 < \zeta \leq n$, where $n = [\zeta] + 1$, (3)

provided that the right hand side is point wise defined on $(\mathfrak{p}, \infty)$."

Definition 3.[13] "CFD of order $\zeta > 0$, for a function $\chi \in \mathcal{C}^{n}((\mathfrak{p}, \infty), \mathbb{R})$, $\mathfrak{p} \geq 0$ is defined as ${}^{C}D^{\zeta}\chi(\mathfrak{z}) = \frac{1}{\Gamma(n-\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z} - s)^{n-\zeta-1}\frac{d^n}{ds^n}\chi(s)ds, n-1 < \zeta \leq n$, where $n = [\zeta] + 1$, (4)

provided that the right hand side is point wise defined on $(\mathfrak{p}, \infty)$."

Definition 4.[10] "The generalized RLFD or HFD of order $\zeta > 0$ and parameter α of a function $\chi \in \mathcal{C}^{n}((\mathfrak{p}, \infty), \mathbb{R})$, $\mathfrak{p} \geq 0$ is defined by ${}^{H}D^{\zeta,\alpha}\chi(\mathfrak{z}) = I^{\alpha(n-\zeta)}D^{n}I^{(1-\alpha)(n-\zeta)}\chi(\mathfrak{z})$,(5)

where $n-1 < \zeta < n$, $0 \leq \alpha \leq 1$, $D = \frac{d}{d\mathfrak{z}}$."

Remark 1. If $\alpha = 0$, then HFD given by definition 4 could be expressed to RLFD and if $\alpha = 1$, then the result will be the emergence of CFD.

Definition 5.[21] "Let E be a Banach space and let $A: E \to E$ be a self map, then A is said to be a nonlinear contraction if there exists a continuous increasing function $\psi: \mathbb{R}^{+} \to \mathbb{R}^{+}$ such that $\psi(0) = 0$ and $\psi(\epsilon) < \epsilon$ for all $\epsilon > 0$ with the property $\| A\chi_1 - A\chi_2 \| \leq \psi(\| \chi_1 - \chi_2 \|)$, for all $\chi_1, \chi_2 \in E$."

Lemma 1.[13]Let $3 < \zeta \leq 4$, $\mathfrak{z} > p$, then $I^{\zeta}\left({}^{RL}D^{\zeta}\chi(\mathfrak{z})\right) = \chi(\mathfrak{z}) - c_1(\mathfrak{z} - \mathfrak{p})^{\zeta-1} - c_2(\mathfrak{z} - \mathfrak{p})^{\zeta-2} - c_3(\mathfrak{z} - \mathfrak{p})^{\zeta-3} - c_4(\mathfrak{z} - \mathfrak{p})^{\zeta-4}$. (6)

3. Qualitative Results

After going through the fundamental ideas illustrated previously, in the next one the authors implemented the theory of fixed points for existence and uniqueness of solutions, which is major goal of our study. In this study, the authors mainly use the notion of compact operator for the existence of solution and the conceptualization of non-linear contraction for EU of solution.

We define the Banach space first to continue the analysis which involves the continuous 4th order derivative functions from $[\mathfrak{p}, \mathfrak{q}] \to \mathbb{R}$ denoted by $\mathcal{C} = \mathcal{C}^{4}([\mathfrak{p}, \mathfrak{q}], \mathbb{R})$ equipped with

$$\| \chi \| = \sup_{t \in [\mathfrak{p}, \mathfrak{q}]} |\chi(\mathfrak{z})|.$$

The motivation of the next lemma is to present the solution of the linear variance of BVP (1) in terms of integral equation.

Lemma 2. Let $\Delta = \gamma(\mathfrak{q} - \mathfrak{p})^{\gamma-1} - k(\tau - \mathfrak{p})^{\gamma} \neq 0$, (7)

then solution of HFD system

$$\begin{cases} {}^{H}D^{\zeta,\alpha}\chi(\mathfrak{z}) + h(\mathfrak{z}) = 0, \mathfrak{z} \in [\mathfrak{p},\mathfrak{q}], \mathfrak{p} \geq 0, \\ \chi(\mathfrak{p}) = \chi'(\mathfrak{p}) = 0, \chi''(\mathfrak{p}) = 0, \chi(\mathfrak{q}) = k\int_{\mathfrak{p}}^{\tau} \chi(s)ds, \end{cases} \quad (8)$$

given as follows: $\chi(\mathfrak{z}) = \begin{cases} (\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{k(\tau-\mathfrak{p})^{\gamma}+\Delta}{\Delta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds - \frac{\gamma k}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}h(s)ds\right) \\ -\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}h(s)ds \end{cases}$ (9)

where $\chi \in C^4([\mathfrak{p},\mathfrak{q}],\mathbb{R})$, $3 < \zeta \leq 4$, $0 \leq \alpha \leq 1$, $\tau \in (\mathfrak{p},\mathfrak{q})$, $k \in \mathbb{R}$, $h: [\mathfrak{p},\mathfrak{q}] \to \mathbb{R}$ is a continuous function, $\gamma = \zeta + 4\alpha - \zeta\alpha$.

Proof. The HFD equation in (8) can be written as

$$I^{\alpha(4-\zeta)}D^4 I^{(1-\alpha)(4-\zeta)}\chi(\mathfrak{z}) + h(\mathfrak{z}) = 0.$$

When both sides are subjected to the RLFI of order ζ, we obtain

$$I^{\zeta}I^{\alpha(4-\zeta)}D^4 I^{(1-\alpha)(4-\zeta)}\chi(\mathfrak{z}) + I^{\zeta}h(\mathfrak{z}) = 0.$$

Besides,

$$I^{\zeta}I^{\alpha(4-\zeta)}D^4 I^{(1-\alpha)(4-\zeta)}\chi(\mathfrak{z}) = I^{\gamma}D^4 I^{(4-\gamma)}\chi(\mathfrak{z}) = I^{\gamma\,RL}D^{\gamma}\chi(\mathfrak{z}),$$

and as a result, we have

$$I^{\gamma\,RL}D^{\gamma}\chi(\mathfrak{z}) + I^{\zeta}h(\mathfrak{z}) = 0.$$

By using Lemma1,we obtain

$$\chi(\mathfrak{z}) = c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-1} + c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-2} + c_3(\mathfrak{z}-\mathfrak{p})^{\gamma-3} + c_4(\mathfrak{z}-\mathfrak{p})^{\gamma-4} - I^{\zeta}h(\mathfrak{z}).$$

We obtain $c_4 = 0$ from the first BC, $\chi(\mathfrak{p}) = 0$, which means

$$\chi(\mathfrak{z}) = c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-1} + c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-2} + c_3(\mathfrak{z}-\mathfrak{p})^{\gamma-3} - I^{\zeta}h(\mathfrak{z}). \quad (10)$$

Now differentiating the equation (10) in order to obtain

$$\chi'(\mathfrak{z}) = (\gamma-1)c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-2} + (\gamma-2)c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-3} + (\gamma-3)c_3(\mathfrak{z}-\mathfrak{p})^{\gamma-4} - I^{\zeta-1}h(\mathfrak{z}).$$

Again following the same procedure as above the second BC $\chi'(\mathfrak{p}) = 0$ gives the value of $c_3 = 0$. Now using the values of constants the solution $\chi(\mathfrak{z})$ becomes

$$\chi(\mathfrak{z}) = c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-1} + c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-2} - I^{\zeta}h(\mathfrak{z}). \quad (11)$$

Now differentiating two times the equation (11) in order to obtain

$$\chi''(\mathfrak{z}) = (\gamma-1)(\gamma-2)c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-3} + (\gamma-2)(\gamma-3)c_2(\mathfrak{z}-\mathfrak{p})^{\gamma-4} - I^{\zeta-2}h(\mathfrak{z}).$$

$c_2 = 0$ is the result of the third BC, $\chi''(\mathfrak{p}) = 0$. By using the values of constants the solution $\chi(\mathfrak{z})$ becomes

$$\chi(\mathfrak{z}) = c_1(\mathfrak{z}-\mathfrak{p})^{\gamma-1} - I^{\zeta}h(\mathfrak{z}). \quad (12)$$

Now the BC: $\chi(\mathfrak{q}) = k\int_{\mathfrak{p}}^{\tau}\chi(s)ds$, gives the value of constant c_1,

$$c_1(\mathfrak{q}-\mathfrak{p})^{\gamma-1} - I^{\zeta}h(\mathfrak{q}) = k\int_{\mathfrak{p}}^{\tau}\chi(s)ds,$$

from which, we get

$$c_1 = \frac{1}{(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}\left(k\int_{\mathfrak{p}}^{\tau}\chi(s)ds + \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds\right).$$

When the value of c_1 in (12) is substituted, we get

$$\chi(\mathfrak{z}) = \frac{1}{(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}\left(k\int_{\mathfrak{p}}^{\tau}\chi(s)ds + \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds\right)(\mathfrak{z}-\mathfrak{p})^{\gamma-1} - \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}h(s)ds.$$

Let

$$A^* = \int_{\mathfrak{p}}^{\tau}\chi(s)ds.$$

So

$$\begin{aligned}
A^* &= \int_{\mathfrak{p}}^{\tau}\chi(\mathfrak{z})d\mathfrak{z} \\
&= \frac{k}{(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}\int_{\mathfrak{p}}^{\tau}\int_{\mathfrak{p}}^{\tau}(\mathfrak{z}-\mathfrak{p})^{\gamma-1}\chi(s)dsd\mathfrak{z} + \frac{1}{(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\tau}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{z}-\mathfrak{p})^{\gamma-1}(\mathfrak{q}-s)^{\zeta-1}h(s)dsd\mathfrak{z} \\
&- \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\tau}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}h(s)dsd\mathfrak{z} \\
&= \frac{k(\tau-\mathfrak{p})^{\gamma}}{\gamma(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}\int_{\mathfrak{p}}^{\tau}\chi(s)ds + \frac{(\tau-\mathfrak{p})^{\gamma}}{\gamma(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds \\
&- \frac{1}{\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}h(s)ds.
\end{aligned}$$

So

$$A^* = \frac{(\tau-\mathfrak{p})^{\gamma}}{\Delta\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds - \frac{\gamma(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}h(s)ds.$$

So

$$\begin{aligned}
\chi(\mathfrak{z}) &= \frac{(\mathfrak{z}-\mathfrak{p})^{\gamma-1}}{(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}\Big(\frac{k(\tau-\mathfrak{p})^{\gamma}}{\Delta\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds \\
&- \frac{k\gamma(\mathfrak{q}-\mathfrak{p})^{\gamma-1}}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}h(s)ds + \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds\Big) - \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}h(s)ds \\
&= (\mathfrak{z}-\mathfrak{p})^{\gamma-1}\Big(\frac{k(\tau-\mathfrak{p})^{\gamma}+\Delta}{\Delta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}h(s)ds \\
&- \frac{k\gamma}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}h(s)ds\Big) - \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}h(s)ds,
\end{aligned}$$

which is the solution as in (9). ▯

Let's define an operator $A: \mathcal{C} \to \mathcal{C}$ by in conjunction with Lemma 2.

$$\begin{aligned}
(A\chi)(\mathfrak{z}) &= (\mathfrak{z}-\mathfrak{p})^{\gamma-1}\Big(\frac{k(\tau-\mathfrak{p})^{\gamma}+\Delta}{\Delta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\wedge(s,\chi(s))ds \\
&- \frac{k\gamma}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}\wedge(s,\chi(s))ds\Big) - \frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\wedge(s,\chi(s))ds.
\end{aligned} \tag{13}$$

Note. As we can see, the solutions to the nonlocal HFD system associated with integral condition (1) constitute the fixed points of the operator A. As a result, examining operator A is sufficient to get the desired outcomes. In contemplating additional analysis, we make the following assumptions.

(H1) The nonlinear function $\wedge$ satisfies

$$|\wedge(\mathfrak{z},\chi_1) - \wedge(\mathfrak{z},\chi_2)| \leq g(t)\frac{|\chi_1-\chi_2|}{\Omega^* + |\chi_1-\chi_2|},$$

for all $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$ and $\chi_1,\chi_2 \in \mathbb{R}$ where the positive constant is defined by

$$\Omega^* = \| g \| \left(\frac{|k|(\tau-\mathfrak{p})^\gamma + |\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1} + \frac{\gamma|k|}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{\zeta+1}(\mathfrak{q}-\mathfrak{p})^{\gamma-1} + \frac{1}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta}\right).$$

and a continuous function $g: [\mathfrak{p},\mathfrak{q}] \to \mathbb{R}^+$.

(H2) There exists a continuous increasing function $\psi: [0,\infty] \to (0,\infty)$ and a function $p \in \mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R}^+)$ such that

$$|\wedge(\mathfrak{z},\chi)| \leq p(\mathfrak{z})\psi(\| \chi \|),$$

for all $(\mathfrak{z},\chi) \in [\mathfrak{p},\mathfrak{q}] \times \mathbb{R}$.

(H3) A positive number M exists such that

$$\frac{M}{\| p \| \psi(M)\Omega} > 1.$$

Theorem 1. (Boyd and Wong fixed point theorem [2]) "Let E be a Banach space and Let $A: E \to E$ be a non linear contraction.Then, A has a unique fixed point on E".

Theorem 2. Let (H1) holds, then the nonlocal HFD system (1) possesses an unique solution on $[\mathfrak{p},\mathfrak{q}]$.

Proof. We define the continuous increasing function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(\epsilon) = \frac{\Omega^*\epsilon}{\Omega^*+\epsilon}$ for all $\epsilon \geq 0$ we have $\psi(0) = 0$ and $\psi(\epsilon) < \epsilon$ for all $\epsilon > 0$. Let $\chi_1,\chi_2 \in \mathcal{C}$, then for all $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$, we have

$$
\begin{aligned}
&|A(\chi_1(\mathfrak{z}))-A(\chi_2(\mathfrak{z}))| \\
&\le (\mathfrak{z}-\mathfrak{p})^{\gamma-1}\Big(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\left|\wedge(s,\chi_1(s))-\wedge(s,\chi_2(s))\right|ds \\
&+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}\left|\wedge(s,\chi_1(s))-\wedge(s,\chi_2(s))\right|ds\Big)+\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}\left|\wedge(s,\chi_1(s))-\wedge(s,\chi_2(s))\right|ds \\
&\le (\mathfrak{z}-\mathfrak{p})^{\gamma-1}\Big(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}g(s)\frac{|\chi_1(s)-\chi_2(s)|}{\Omega^*+|\chi_1(s)-\chi_2(s)|}ds \\
&+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}g(s)\frac{|\chi_1(s)-\chi_2(s)|}{\Omega^*+|\chi_1(s)-\chi_2(s)|}ds\Big)+\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}g(s)\frac{|\chi_1(s)-\chi_2(s)|}{\Omega^*+|\chi_1(s)-\chi_2(s)|}ds \\
&\le \|g\|\frac{\psi(\|\chi_1-\chi_2\|)}{\Omega^*}\Big((\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}ds\right) \\
&+\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}ds\Big) \\
&=\|g\|\frac{\psi(\|\chi_1-\chi_2\|)}{\Omega^*}\Big((\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta}+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{(\zeta+1)}\right) \\
&+\frac{1}{\Gamma(\zeta+1)}(\mathfrak{z}-\mathfrak{p})^{\zeta}\Big) \\
&\le \|g\|\frac{\psi(\|\chi_1-\chi_2\|)}{\Omega^*}\Big(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\gamma-1} \\
&+\frac{1}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta}\Big) \\
&=\frac{\psi(\|\chi_1-\chi_2\|)}{\Omega^*}\Omega^* \\
&=\psi(\|\chi_1-\chi_2\|).
\end{aligned}
$$

Which gives that $\| A(\chi_1)-A(\chi_2) \| \le \psi(\| \chi_1-\chi_2 \|)$.

A is a nonlinear contraction as an outcome. As it turns out, according to the Boyd and Wong fixed point theorem, operator A has a single fixed point which gives the solution to nonlocal HFD system. □

Theorem 3. (LeraySchauder Non-linear Alternative for Single Valued Maps[9] .) “Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $A: \overline{U} \to C$ is a continuous, compact map(that is, $A(\overline{U})$ is a relatively compact subset of C). Then either

1. A has a fixed point in $\overline{U}$,or
2. There is a $\chi \in \partial U$(the boundary of U in C) and $\lambda \in (0,1)$ with $\chi = \lambda A(\chi)$."

Theorem 4. Let $\wedge: [\mathfrak{p},\mathfrak{q}] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and (H2) and (H3) hold, then the nonlocal HFD system (1) possesses atleast one solution on $[\mathfrak{p},\mathfrak{q}]$.

Proof. To find fixed point of A, we will employ Leray Schauder Non-linear Alternative for Single Valued Maps. We define the positive constant Ω by

$$
\Omega=\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}+\frac{\gamma|k|}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{\zeta+1}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}+\frac{1}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta}\right).
$$

The proof has been presented in phases..
Step 1. A is continuous:

Let us consider the the convergent sequence $\chi_n \to \chi$ in $\mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R})$. Next, for each $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$,

$$|A(\chi_n(\mathfrak{z})) - A(\chi(\mathfrak{z}))|$$
$$\leq (\mathfrak{z}-\mathfrak{p})^{\gamma-1}(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}|\wedge(s,\chi_n(s))-\wedge(s,\chi(s))|ds$$
$$+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}|\wedge(s,\chi_n(s))-\wedge(s,\chi(s))|ds)+\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}|\wedge(s,\chi_n(s))-\wedge(s,\chi(s))|ds$$
$$\leq\|\wedge(.,\chi_n(.))-\wedge(.,\chi(.))\|((\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}ds+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}ds\right)$$
$$+\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}ds)$$
$$=\|\wedge(.,\chi_n(.))-\wedge(.,\chi(.))\|((\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta}+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{(\zeta+1)}\right)$$
$$+\frac{1}{\Gamma(\zeta+1)}(\mathfrak{z}-\mathfrak{p})^{\zeta})$$
$$\leq\|\wedge(.,\chi_n(.))-\wedge(.,\chi(.))\|(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}$$
$$+\frac{1}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta})$$
$$=\|\wedge(.,\chi_n(.))-\wedge(.,\chi(.))\|\Omega.$$

Which implies that $\| A(\chi_n)-A(\chi)\|\leq\Omega\|\wedge(.,\chi_n(.))-\wedge(.,\chi(.))\|$. Also, the continuity of $\wedge$ implies $||A(\chi_n)-A(\chi)||\to 0$as $n\to\infty$.

Step 2. In $\mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R})A$ transforms bounded sets into bounded sets. Take any $r>0$, we define $B_r=\{\chi\in\mathcal{C}:\|\chi\|\leq r\}$ by (H2) for each $\chi\in B_r$ and for each $\mathfrak{z}\in[\mathfrak{p},\mathfrak{q}]$, Let $l=\|p\|\psi(\|\chi\|)\Omega>0$, we have

$$|(A\chi)(\mathfrak{z})|$$
$$\leq(\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}|\wedge(s,\chi(s))|ds+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}|\wedge(s,\chi(s))|ds\right)$$
$$+\frac{1}{\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{z}}(\mathfrak{z}-s)^{\zeta-1}|\wedge(s,\chi(s))|ds$$
$$\leq\|p\|\psi(\|\chi\|)((\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta}+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{\zeta+1}\right)$$
$$+\frac{1}{\Gamma(\zeta+1)}(\mathfrak{z}-\mathfrak{p})^{\zeta})$$
$$\leq\|p\|\psi(\|\chi\|)((\mathfrak{z}-\mathfrak{p})^{\gamma-1}\left(\frac{|k|(\tau-\mathfrak{p})^{\gamma}+|\Delta|}{|\Delta|(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta+\gamma-1}+\frac{|k|\gamma}{|\Delta|\Gamma(\zeta+2)}(\tau-\mathfrak{p})^{\zeta+1}(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\right)$$
$$+\frac{1}{\Gamma(\zeta+1)}(\mathfrak{q}-\mathfrak{p})^{\zeta})$$
$$=\|p\|\psi(\|\chi\|)\Omega$$
$$=l.$$

So there exists $l>0$ such that $\| A(\chi)\|\leq l$. Therefore $AB_r\subseteq B_l$. Hence the claim made in step 2 is proved.

Step 3.A maps bounded sets into equicontinuous sets of $\mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R})$.
Let $t_1,t_2\in[\mathfrak{p},\mathfrak{q}],t_1<t_2$, B_r be a bounded set of $\mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ as in step 2, and let $\chi\in B_r$, then

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$$
\begin{aligned}
&|(A\chi)(\mathrm{t}_2)-(A\chi)(\mathrm{t}_1)| \\
&= |((\mathrm{t}_2-\mathfrak{p})^{\gamma-1}-(\mathrm{t}_1-\mathfrak{p})^{\gamma-1}) \\
&\quad \times\left(\frac{k(\tau-\mathfrak{p})^{\gamma}+\Delta}{\Delta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\wedge(s,\chi(s))\,ds\right. \\
&\quad -\frac{k\gamma}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}\wedge(s,\chi(s))\,ds \\
&\quad -\frac{1}{\Gamma(\zeta)}\left(\int_{\mathfrak{p}}^{\mathrm{t}_2}(\mathrm{t}_2-s)^{\zeta-1}\wedge(s,\chi(s))\,ds-\int_{\mathfrak{p}}^{\mathrm{t}_1}(\mathrm{t}_1-s)^{\zeta-1}\wedge(s,\chi(s))\,ds\right. \\
&\quad +\int_{\mathfrak{p}}^{\mathrm{t}_1}(\mathrm{t}_2-s)^{\zeta-1}\wedge(s,\chi(s))\,ds-\int_{\mathfrak{p}}^{\mathrm{t}_1}(\mathrm{t}_2-s)^{\zeta-1}\wedge(s,\chi(s))\,ds \\
&= |((\mathrm{t}_2-\mathfrak{p})^{\gamma-1}-(\mathrm{t}_1-\mathfrak{p})^{\gamma-1}) \\
&\quad \times\left(\frac{k(\tau-\mathfrak{p})^{\gamma}+\Delta}{\Delta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\wedge(s,\chi(s))\,ds\right. \\
&\quad -\frac{k\gamma}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}\wedge(s,\chi(s))\,ds \\
&\quad -\frac{1}{\Gamma(\zeta)}\left(\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-s)^{\zeta-1}\wedge(s,\chi(s))\,ds-\int_{\mathfrak{p}}^{\mathrm{t}_1}(\mathrm{t}_1-s)^{\zeta-1}\wedge(s,\chi(s))\,ds\right. \\
&\quad +\int_{\mathfrak{p}}^{\mathrm{t}_1}(\mathrm{t}_2-s)^{\zeta-1}\wedge(s,\chi(s))\,ds \\
&= |((\mathrm{t}_2-\mathfrak{p})^{\gamma-1}-(\mathrm{t}_1-\mathfrak{p})^{\gamma-1}) \\
&\quad \times\left(\frac{k(\tau-\mathfrak{p})^{\gamma}+\Delta}{\Delta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\wedge(s,\chi(s))\,ds\right. \\
&\quad -\frac{k\gamma}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}\wedge(s,\chi(s))\,ds \\
&-\frac{1}{\Gamma(\zeta)}\left(\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-s)^{\zeta-1}\wedge(s,\chi(s))\,ds\right. \\
&\quad +\int_{\mathfrak{p}}^{\mathrm{t}_1}((\mathrm{t}_1-s)^{\zeta-1}-(\mathrm{t}_2-s)^{\zeta-1})\wedge(s,\chi(s))\,ds \\
&= \| p \| \psi(\| \chi \|)(((\mathrm{t}_2-\mathfrak{p})^{\gamma-1}-(\mathrm{t}_1-\mathfrak{p})^{\gamma-1}) \\
&\quad \times\left(\frac{k(\tau-\mathfrak{p})^{\gamma}+\Delta}{\Delta(\mathfrak{q}-\mathfrak{p})^{\gamma-1}\Gamma(\zeta)}\int_{\mathfrak{p}}^{\mathfrak{q}}(\mathfrak{q}-s)^{\zeta-1}\,ds\right. \\
&\quad +\frac{k\gamma}{\Delta\Gamma(\zeta+1)}\int_{\mathfrak{p}}^{\tau}(\tau-s)^{\zeta}\,ds \\
&+\frac{1}{\Gamma(\zeta)}\left(\int_{\mathrm{t}_1}^{\mathrm{t}_2}(\mathrm{t}_2-s)^{\zeta-1}\,ds\right. \\
&\quad +\int_{\mathfrak{p}}^{\mathrm{t}_1}((\mathrm{t}_1-s)^{\zeta-1}-(\mathrm{t}_2-s)^{\zeta-1})\,ds \\
&\quad).
\end{aligned}
$$

Hence, $\| (A\chi)(\mathrm{t}_2)-(A\chi)(\mathrm{t}_1) \|$ tends to zero as $\mathrm{t}_1 \to \mathrm{t}_2$. In light of the Arzela-Ascoli theorem and steps 1 through 3, it could easily be concluded that A is continuous, completely continuous, which makes it compact.

Step 4. Now we shall prove that there exists an open set $U \subset \mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ s.t there is no $\chi \in \partial U$ with $\chi = \lambda(A\chi)$, for $0 < \lambda < 1$.

Let $\chi \in \mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R})$ be such that $\chi = \lambda(A\chi)$ for some $0 < \lambda < 1$.Then for each $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$,we have (from step2)

$$\begin{aligned} &|\chi(\mathfrak{z})| \\ &= \lambda|(A\chi)(\mathfrak{z})| \\ &\leq |(A\chi)(\mathfrak{z})| \\ &\leq \| p \| \psi(\| \chi \|)\Omega. \end{aligned}$$

therefore

$$\frac{\| \chi \|}{\| p \| \psi(\| \chi \|)\Omega} \leq 1.$$

So (By H3)there exists $M > 0$ s.t $\| \chi \| \neq M$.

Let's establish $U = \{\chi \in \mathcal{C}([\mathfrak{p},\mathfrak{q}],\mathbb{R}): \| \chi \| < M\}$. Thus, there is no $\chi \in \partial U$ with $\chi = \lambda(A\chi)$, $0 < \lambda < 1$. Hence by Leray Schauder non-linear alternative, A possesses at least one fixed point $\chi \in \overline{U}$, which gives the solution of nonlocal HFD system (1). ▯

4. Illustrative Examples

Example1.Consider the nonlocal fractional HFD system

$$\begin{cases} {}^{H}D^{\frac{7}{2},\frac{1}{2}}\chi(\mathfrak{z}) = \frac{1}{\mathfrak{z}+4}\left(\frac{|\chi(\mathfrak{z})|}{1+|\chi(\mathfrak{z})|}\right) + \frac{2}{3}, \mathfrak{z} \in [0,1], \\ \chi(0) = \chi'(0) = 0, \chi''(0) = 0, \chi(1) = \frac{2}{3}\int_0^{\frac{1}{2}} \chi(s)ds, \end{cases} \qquad (14)$$

Here $\zeta = \frac{7}{2}, \alpha = \frac{1}{2}, \gamma = \frac{15}{4}, \mathfrak{p} = 0, \mathfrak{q} = 1, \tau = \frac{1}{2}, k = \frac{2}{3}$. We consider $g(\mathfrak{z}) = \frac{1}{\mathfrak{z}+4}$,$\| g \| = \frac{1}{5}$ and by direct calculation $\Omega^* = 0.00802$ for all $\mathfrak{z} \in [\mathfrak{p},\mathfrak{q}]$ and $\chi_1, \chi_2 \in \mathbb{R}$, then

$$\begin{aligned} &|\wedge(\mathfrak{z},\chi_1) - \wedge(\mathfrak{z},\chi_2)| \\ &\leq \frac{1}{\mathfrak{z}+4}\left(\frac{|\chi_1(\mathfrak{z}) - \chi_2(\mathfrak{z}|}{1+|\chi_1(\mathfrak{z}) - \chi_2(\mathfrak{z}|}\right) \\ &\leq g(\mathfrak{z})\left(\frac{|\chi_1(\mathfrak{z}) - \chi_2(\mathfrak{z}|}{\Omega^*+|\chi_1(\mathfrak{z}) - \chi_2(\mathfrak{z}|}\right). \end{aligned}$$

So (H1) is satisfied.Hence by theorem2,BVP(14) has unique solution on [0,1].

Example 2. If the non linear function $\wedge(\mathfrak{z},\chi)$ in (14) is considered as

$$\wedge(\mathfrak{z},\chi) = \frac{1}{\mathfrak{z}+4}\left(\frac{\chi^4}{1+\chi^2} + \frac{1}{2}\right). \qquad (15)$$

Here

$$\begin{aligned} &|\wedge(\mathfrak{z},\chi)| \\ &\leq \frac{1}{\mathfrak{z}+4}\left(\chi^2 + \frac{1}{2}\right) \\ &\leq p(\mathfrak{z})\psi(\| \chi \|), \end{aligned}$$

for each$(\mathfrak{z},\chi) \in [\mathfrak{p},\mathfrak{q}] \times \mathbb{R}$, if we choose $p(\mathfrak{z}) = \frac{1}{\mathfrak{z}+4}$, $\psi(\chi) = \chi^2 + \frac{1}{2}$ and by choosing a constant $M = 99.6 > 0$ such that $\frac{M}{\|p\|\psi(M)\Omega} = 1.0014577696 > 1$,then (H2) is satisfied.Hence by theorem4, BVP (14)with $\wedge$ given by (15) has atleast one solution on [0,1].

5. Conclusions

In our work, we have established EU results for solution of nonlocal fractional HFD system (1)by converting it into fixed point operator and applied fixed point theorem by Boyd and Wong on nonlinear contractions and LeraySchauder on nonlinear alternative for single valued maps . We have covered case of our studied system $3 < \zeta < 4$. Our established results are in new configuration and enrich the literature. Examples are also have been presented to justify the results. In the future these results can be generalized to cover the more cases for ζ, inclusion problems with multipoint BC can be considered including Hilfer, Hilfer-Hadamard fractional derivative.

Data Availability

No data were used to support this study.

Conflict of interest

The authors declare that they have no competing interests.

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